

A HYPERKÄHLER SUBMANIFOLD OF THE MONOPOLE MODULI SPACE

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ABSTRACT. We discuss a $4[k/2]$ -dimensional complete hyperkähler submanifold of the $(4k - 4)$ -dimensional moduli space of strongly centred $SU(2)$ -monopoles of charge k .

The isometry group of \mathbb{R}^3 acts isometrically on the moduli space M_k of Euclidean $SU(2)$ -monopoles of charge k and, consequently, a fixed point set of any subgroup of this group is a totally geodesic submanifold. Similar statement holds for any subgroup of the orthogonal group $O(3)$ acting on the submanifold M_k^0 consisting of *strongly centred* monopoles [8] (these are monopoles with the centre at the origin and total phase equal to 1). Houghton and Sutcliffe [9] have shown that the submanifold of M_3^0 consisting of monopoles symmetric about the origin is isometric to the Atiyah-Hitchin manifold (i.e. the moduli space of centred monopoles of charge 2). Surprisingly, monopoles of higher charges invariant under the reflection $x \mapsto -x$ seem not to have been considered in the literature. This reflection is particularly interesting since it extends to a reflection $\tau : (x, t) \mapsto (-x, t^{-1})$ on $\mathbb{R}^3 \times S^1$ which preserves the hyperkähler structure of $\mathbb{R}^3 \times S^1$. It is then easy to deduce that the submanifold N_k of strongly centred monopoles symmetric about the origin is a (complete) hyperkähler submanifold of M_k^0 for any charge k . We have stumbled upon this hyperkähler manifold (for even k) in a completely different context in [3] and realised only *a posteriori*, by identifying the twistor space, that it must be a submanifold of M_k^0 .

In the present paper we describe the submanifold N_k in terms of Nahm's equations. Since N_k is $SO(3)$ -invariant, all of its complex structures are equivalent and can be identified with a complex submanifold of based rational maps of degree k . This is straightforward, given that the involution τ acts on rational maps via $p(z)/q(z) \mapsto \tilde{p}(z)/q(-z)$, where $\tilde{p}(-z)p(z) - 1 = 0 \pmod{q(z)}$, but we also show this directly using Nahm's equations. We then show that N_k is biholomorphic to the *transverse Hilbert scheme* of n points [3, §5] on the D_1 -surface if $k = 2n$, and on the D_0 -surface if $k = 2n + 1$. This allows us to conclude that N_{2n} is simply connected, while N_{2n+1} has fundamental group of order 2. In Section 2 we present an alternative construction, which also gives a description of hyperkähler deformations of N_{2n} .

1. DESCRIPTION IN TERMS OF NAHM'S EQUATIONS

The moduli space M_k^0 of strongly centred $SU(2)$ -monopoles of charge k is isomorphic to the moduli space of $\mathfrak{su}(k)$ -valued solutions to Nahm's equations on $(0, 2)$ with $T_1(t), T_2(t), T_3(t)$ having simple poles at $t = 0, 2$, the residues of which define

the standard k -dimensional irreducible representation of $\mathfrak{su}(2)$, i.e.

$$\text{Res } T_1(t) = i \text{diag}(k-1, k-3, \dots, 3-k, 1-k),$$

$$\text{Res}(T_2 + iT_3)(t)_{ij} = \begin{cases} \sqrt{j(k-j)} & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding representation of $\mathfrak{sl}_2(\mathbb{C})$ is given by the action $y\partial_x, x\partial_y, x\partial_x - y\partial_y$ on binary forms of degree $k-1$ with a basis

$$(1.1) \quad v_i = \binom{k-1}{i-1}^{1/2} x^{k-i} y^{i-1}, \quad i = 1, \dots, k.$$

Write V for the standard 2-dimensional $\mathfrak{sl}_2(\mathbb{C})$ -module, so that the symmetric product $S^{k-1}V$ is the standard k -dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. The equivariant isomorphisms

$$\Lambda^2(S^{2n+1}V^*) \simeq \bigoplus_{i=0}^n S^{4i}V^*, \quad S^2(S^{2n}V^*) \simeq \bigoplus_{i=0}^n S^{4i}V^*,$$

imply that, if k is even (resp. if k is odd), then there exists a unique (up to scaling) $\mathfrak{sl}_2(\mathbb{C})$ -invariant skew-symmetric (resp. symmetric) bilinear form on $S^{k-1}V$. This invariant form is a classical object and is called *transvectant*. In the basis (1.1) the transvectant $T(f, g)$ of $f = \sum_{i=1}^k a_i v_i$, $g = \sum_{i=1}^k b_i v_i$ is given [10, Ex.2.7]

$$T(f, g) = \sum_{i=0}^{k-1} (-1)^i a_{i+1} b_{k-i}.$$

We conclude that the residues of T_1, T_2, T_3 belong to a symplectic subalgebra of $\mathfrak{su}(k)$ if k is even, and to an orthogonal subalgebra if k is odd. These subalgebras are defined as

$$(1.2) \quad \{A \in \mathfrak{su}(k); AJ + JA^T = 0\},$$

where J is an antidiagonal matrix with $J_{i, k+1-i} = (-1)^{i-1}$. In other words, they are the fixed point sets of the involution

$$(1.3) \quad \sigma(A) = -JA^T J^{-1}.$$

We denote by $\mathfrak{su}(k)^\sigma$ the subalgebra (1.2) and by $SU(k)^\sigma$ the corresponding subgroup of $SU(k)$ ($SU(k)^\sigma \simeq Sp(n)$ if $k = 2n$ and $SU(k)^\sigma \simeq SO(2n+1)$ if $k = 2n+1$).

We consider the space \mathcal{A}^σ of $\mathfrak{su}(k)^\sigma$ -valued solutions to Nahm's equations on $[0, 2]$. Let \mathcal{G} denote the group of $SU(k)$ -valued gauge transformations which are identity at $t = 0, 2$, and let \mathcal{G}^σ be its subgroup of $SU(k)^\sigma$ -valued gauge transformations. It is easy to verify that two \mathcal{G} -equivalent elements of \mathcal{A}^σ are also \mathcal{G}^σ -equivalent. Thus the natural map $\mathcal{A}^\sigma / \mathcal{G}^\sigma \rightarrow M_k^0$ is an embedding and we view $N_k = \mathcal{A}^\sigma / \mathcal{G}^\sigma$ as a submanifold of M_k^0 . N_k is the fixed point set of an involution σ which sends each $T_i(t)$ to $\sigma(T_i(t))$ (and acts the same way on gauge transformations) and therefore a complete hyperkähler submanifold of M_k^0 .

Proposition 1.1. *With respect to any complex structure N_k is biholomorphic to the space of based rational maps $\frac{p(z)}{q(z)}$ of degree k such that*

- (i) *if $k = 2n$, then $q(z) = \tilde{q}(z^2)$ for a monic polynomial \tilde{q} of degree n and $p(z)p(-z) \equiv 1 \pmod{q(z)}$;*

- (ii) if $k = 2n + 1$, then $q(z) = z\tilde{q}(z^2)$ for a monic polynomial \tilde{q} of degree n , $p(0) = 1$ and $p(z)p(-z) \equiv 1 \pmod{q(z)}$.

In particular $\dim_{\mathbb{R}} N_{2n} = \dim_{\mathbb{R}} N_{2n+1} = 4n$.

Remark 1.2. If the roots of $q(z)$ are distinct, then the above condition means that $p(w)p(-w) = 1$ for each root w of $q(z)$ (and $p(0) = 1$ if k is odd). The full N_k is then the closure of this set inside the space of all rational maps.

Proof. Recall [6] that M_k is biholomorphic to the space of based (i.e. $f(\infty) = 0$) rational maps of degree k , and M_k^0 to the submanifold consisting of rational maps $p(z)/q(z)$ such that the sum of the poles z_i is equal to 0 and $\prod_{i=1}^k p(z_i) = 1$. From the point of view of Nahm's equations, the rational map is obtained by applying a (singular) complex gauge transformation g to the complex Nahm equation $\beta = [\beta, \alpha]$ in order to make β a constant matrix of the form

$$(1.4) \quad S = \begin{pmatrix} 0 & \dots & \dots & 0 & s_n \\ 1 & \ddots & & 0 & s_{n-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & \ddots & 0 & s_2 \\ 0 & \dots & \dots & 1 & s_1 \end{pmatrix}.$$

The value of the complex gauge transformation $g(t)$ at $t = 2$ (modulo a fixed singular gauge transformation) is then an element u of the centraliser of S in $GL(k, \mathbb{C})$. The pair (S, u) corresponds to the rational map $\text{tr } u(z - S)^{-1}$ (this is the description given in [2]). The complex structures of M_k^0 and of N_k are obtained by assuming that S and u belong to the appropriate subalgebra and subgroup (i.e. to $\mathfrak{sl}(k, \mathbb{C}), SL(n, \mathbb{C})$ for M_k^0 and to $\mathfrak{sl}(k, \mathbb{C})^\sigma, SL(n, \mathbb{C})^\sigma$ for N_k). It is enough to consider the subset where the poles of the rational map, i.e. the eigenvalues of S , are distinct (since N_k is the closure of this set in the space of all rational maps). A Cartan subalgebra \mathfrak{h} of $\mathfrak{sl}(k, \mathbb{C})^\sigma$ is given by the diagonal matrices h satisfying $h_{ii} + h_{k+1-i, k+1-i} = 0$, $i = 1, \dots, k$. It is then immediate that if (S, u) is conjugate to an element of $\mathfrak{h} \times H$ ($H = \exp \mathfrak{h}$), then the corresponding rational map satisfies the conditions in the statement. \square

In [3, Ex. 5.4] we have identified the complex manifold described in the above proposition for $k = 2n$ as the *Hilbert scheme of n points on the D_1 -surface $x^2 - zy^2 = 1$ transverse to the projection $(x, y, z) \mapsto z$* (similarly, the space of all rational maps of degree k is the Hilbert scheme of k points on $\mathbb{C}^* \times \mathbb{C}$ transverse to the projection onto the second factor [1, Ch. 6]). It turns out that for odd k the complex structure of N_k is that of the transverse Hilbert scheme of points on the D_0 -surface $x^2 - zy^2 - y = 0$:

- Proposition 1.3.** (i) *With respect to any complex structure N_{2n} is biholomorphic to the Hilbert scheme of n points on the D_1 -surface $x^2 - zy^2 = 1$ transverse to the projection $(x, y, z) \mapsto z$.*
- (ii) *With respect to any complex structure N_{2n+1} is biholomorphic to the Hilbert scheme of n points on the D_0 -surface $x^2 - zy^2 - y = 0$ transverse to the projection $(x, y, z) \mapsto z$.*

Remark 1.4. The fact that N_3 is biholomorphic to the D_0 -surface has been observed by Houghton and Sutcliffe [9].

Proof. Part (i) has already been shown in [3, Ex. 5.4]. For part (ii) recall from [3] that the Hilbert scheme of n points on the D_0 -surface $x^2 - zy^2 + y = 0$ transverse to the projection $(x, y, z) \mapsto z$ is an affine variety in \mathbb{C}^{3n} given by the same equation, but for polynomials. More precisely its points are polynomials $x(z), y(z), r(z)$ with degrees of x and y at most $n-1$ and $r(z)$ a monic polynomial of degree n , satisfying the condition

$$x(z)^2 - zy(z)^2 - y(z) = 0 \pmod{r(z)}.$$

As in [3] write $z = u^2$ and rewrite the above equation as

$$(x(u^2) + uy(u^2))(x(u^2) - uy(u^2)) - \frac{(x(u^2) + uy(u^2)) - (x(u^2) - uy(u^2))}{u} = 0 \pmod{r(u^2)}.$$

Let us write $f(u) = x(u^2) + uy(u^2)$, $p(u) = 1 + uf(u)$, $q(u) = ur(u^2)$. Then it is easy to see that if the roots of q are distinct, then the last equation is equivalent to the condition $p(w)p(-w) = 1$ for any nonzero root w of $q(u)$. Since this last equation is polynomial in the coefficients of p and q , it describes a closed affine subvariety inside the affine variety of all rational maps. Since the two closed affine subvarieties, namely the transverse Hilbert scheme of points on the D_0 -surface and the variety described in Proposition 1.1, have a common open dense subset, they must coincide. \square

We can now compute the fundamental group of N_k :

Proposition 1.5.

$$\pi_1(N_k) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ \mathbb{Z}_2 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The fundamental groups of the D_0 - and D_1 -surface (i.e. the Atiyah-Hitchin manifold and its double cover) are well-known [1] and equal to \mathbb{Z}_2 and to 1 respectively. The result follows from

Lemma 1.6. *Let X be a smooth complex surface and $\pi : X \rightarrow C$ a holomorphic submersion onto a connected Riemann surface C . Suppose further that, over an open dense subset of C , π is a locally trivial fibration with connected fibres. Then the fundamental group of the transverse Hilbert scheme $X_\pi^{[n]}$ of n points, $n \geq 2$, is equal to $H_1(X, \mathbb{Z})$.*

We first observe that if X is a smooth complex surface and $n \geq 2$, then the fundamental group of the full Hilbert scheme $X^{[n]}$ of n points, $n \geq 2$, is equal to $H_1(X, \mathbb{Z})$. This follows by combining two facts: 1) a classical result of Dold and Puppe [5, Thm. 12.15] which says that $\pi_1(S^n X) \simeq H_1(X, \mathbb{Z})$ for $n \geq 2$, and 2) a theorem of Kollár [11, Thm. 7.8] which implies that $\pi_1(X^{[n]}) \simeq \pi_1(S^n X)$.

We now aim to show that $\pi_1(X_\pi^{[n]}) \simeq \pi_1(X^{[n]})$. Let Y denote a submanifold of $X^{[n]}$ consisting of $D \in X^{[n]}$ with $\pi(D) = z_0 + E$, where E has length $n-2$ in C , $z_0 \notin \text{supp } E$, and $D \cap \pi^{-1}(z_0)$ consists of two distinct points. Clearly $Y \cap X_\pi^{[n]} = \emptyset$. Denote by U a tubular neighbourhood of Y and let $Z = U \cup X_\pi^{[n]}$, $W = U \cap X_\pi^{[n]}$. Since the complement of Z in $X^{[n]}$ has (complex) codimension 2, the fundamental groups of Z and of $X^{[n]}$ coincide. Since W is a punctured disc bundle over Y , the long exact sequence of homotopy groups implies that the map $\pi_1(W) \rightarrow \pi_1(U)$ is surjective and its kernel consists of at most the “meridian loop” around Y . Owing

to the assumption, we can choose a point y of Y so that a neighbourhood of the fibre containing two points is isomorphic to $B \times F$, where B is a disc $\{z \in \mathbb{C}; |z| < 1 + \epsilon\}$ and F is the generic fibre of π . The intersection of a neighbourhood of y in Z with W is then of the form $S_\pi^2(B \times F) \times V$, where the subscript π means that the pair of points $\{x_1, x_2\} \in B \times F$ satisfies $\pi(x_1) \neq \pi(x_2)$ and V is an open subset in $X_\pi^{[n-2]}$. Let ρ be the projection $(B \times F) \times (B \times F) \rightarrow S^2(B \times F)$. A “meridian loop” in W around Y can be then chosen to be

$$t \mapsto (\rho(e^{\pi it}, f, e^{-\pi it}, f), v), \quad t \in [0, 1]$$

for constant f and v . This loop is contractible in $X_\pi^{[n]}$: the homotopy

$$H(r, t) = (\rho(re^{\pi it}, f, re^{-\pi it}, f), v), \quad t, r \in [0, 1]$$

contracts it to (D, v) where D is the double point $(\{z^2 = 0\}, f)$ in $B \times F$. To recapitulate: we have shown that the map $\pi_1(W) \rightarrow \pi_1(U)$ is surjective and its kernel has trivial image in $\pi_1(X_\pi^{[n]})$. It follows that the amalgamated free product $\pi_1(U) *_{\pi_1(W)} \pi_1(X_\pi^{[n]})$ is isomorphic to $\pi_1(X_\pi^{[n]})$ and, hence, van Kampen’s theorem implies that $\pi_1(Z) \simeq \pi_1(X_\pi^{[n]})$. \square

Remark 1.7. The D_0 -surface X is the quotient of the D_1 -surface \tilde{X} by a free action of \mathbb{Z}_2 given by $(x, y, z) \mapsto (-x, -y, z)$. This induces a free \mathbb{Z}_2 -action on the transverse Hilbert scheme $\tilde{X}_\pi^{[n]}$ of n points for any n , but $\tilde{X}_\pi^{[n]}/\mathbb{Z}_2 \not\simeq X_\pi^{[n]}$, unless $n = 1$. Certainly, there is a surjective holomorphic map $\tilde{X}_\pi^{[n]} \rightarrow X_\pi^{[n]}$ for any n , which, in the description of these spaces provided in Proposition 1.1, sends a rational function $p(z)/q(z) \in \tilde{X}_\pi^{[n]}$ to $\bar{p}(z)/zq(z)$, where $\bar{p}(z) = p(z)^2 \bmod zq(z)$. This map is constant on \mathbb{Z}_2 -orbits, but it is generically 2^n -to-1 and not a covering (the preimage of a point consists of 2^m points, where $2m$ is the number of distinct roots of $q(z)$).

Remark 1.8. It is instructive to compare spectral curves of monopoles in N_{2n+1} to those in N_{2n} . It follows from Proposition 1.1 that the spectral curve S of a monopole in N_{2n+1} is always singular and given by an equation of the form $\eta P(\zeta, \eta) = 0$, where ζ is the affine coordinate of \mathbb{P}^1 , η is the induced fibre coordinate in $T\mathbb{P}^1$, and P a polynomial of the form $\eta^{2n} + \sum_{i=1}^n a_i(\zeta)\eta^{2n-2i}$, $\deg a_i(\zeta) = 4i$. A spectral curve of a monopole satisfies [1] the condition $L^2|_S \simeq \mathcal{O}$, where L^2 is a line bundle on $T\mathbb{P}^1$ with transition function $\exp(2\eta/\zeta)$. It follows that the line bundle L^2 is also trivial on the curve \tilde{S} defined by the equation $P(\zeta, \eta) = 0$. Conversely, the spectral curve \tilde{S} of a monopole in N_{2n} admits a section $s(\zeta, \eta)$ of L^2 which satisfies $s(\zeta, \eta)s(\zeta, -\eta) \equiv 1 \bmod P(\zeta, \eta)$, where $P(\zeta, \eta) = 0$ is the equation of \tilde{S} . It follows that $s(\zeta, 0) = \pm 1$ and if we set $\bar{s}(\zeta, \eta) = s^2(\zeta, \eta)$ on \tilde{S} and $\bar{s}(\zeta, \eta) \equiv 1$ on $\eta = 0$, we obtain a nonvanishing section of L^4 on the curve $\eta P(\eta, \zeta) = 0$, i.e. a section of L^2 on the curve S given by $\tilde{\eta}P(\tilde{\eta}/2, \zeta) = 0$, where $\tilde{\eta} = 2\eta$. In the case $n = 1$, Houghton and Sutcliffe [9] have shown that for $n = 1$ these maps $S \mapsto \tilde{S}$ and $\tilde{S} \mapsto S$ send spectral curves of monopoles in N_3 to spectral curves of monopoles in N_2 and vice versa¹, but for higher n this is not the case. The reason is that Hitchin’s [7] nonsingularity condition $H^0(S, L^t(k-2)) = 0$, $t \in (0, 2)$, is not necessarily satisfied for the resulting curves.

¹Strictly speaking, the curve \tilde{S} obtained from S must be rescaled via $\eta = 2\tilde{\eta}$ in order to be the spectral curve of a monopole in N_2 .

2. DEFORMATIONS AND COVERINGS

Dancer [4] has shown that the D_1 -surface admits a 1-parameter family of deformations carrying complete hyperkähler metrics. As we observed in [3], the transverse Hilbert schemes of points on these deformations also admit natural complete hyperkähler metrics. We wish to describe these metrics as deformations of manifolds N_{2n} . We begin by describing N_k without reference to an embedding into M_k^0 .

Let G_k (resp. \mathfrak{g}_k) denote $Sp(n)$ (resp. $\mathfrak{sp}(n)$) if $k = 2n$ and $SO(2n+1)$ (resp. $\mathfrak{so}(k)$) if $k = 2n+1$. The construction of the previous section shows that N_k is the moduli space of \mathfrak{g}_k -valued solutions to Nahm's equations on $(0, 2)$ with simple poles at $t = 0, 2$ and residues defining the principal homomorphism $\mathfrak{su}(2) \rightarrow \mathfrak{g}_k$, modulo G_k -valued gauge transformations which are identity at $t = 0, 2$. This moduli space is, in turn, a finite-dimensional hyperkähler quotient of a simpler hyperkähler manifold. Let W_k^- (resp. W_k^+) be the moduli space of \mathfrak{g}_k -valued solutions to Nahm's equations on $(0, 1]$ (resp. $[1, 2)$) with the above boundary behaviour at $t = 0$ (resp. at $t = 2$) and regular at $t = 1$, modulo G_k -valued gauge transformations which are identity at $t = 0, 1$ (resp. at $t = 1, 2$). W_k^\pm are hyperkähler manifolds (biholomorphic to $G_k^\mathbb{C} \times \mathbb{C}^n$ [2]) with an isometric and triholomorphic action of G_k obtained by allowing gauge transformations with an arbitrary value at $t = 1$. Then N_k is the hyperkähler quotient of $W_k^- \times W_k^+$ by the diagonal G_k .

We can also describe in a similar manner the universal (i.e. double) covering space of N_{2n+1} : it is given by the same construction, but with $G_{2n+1} = Spin(2n+1)$ instead of $SO(2n+1)$.

An alternative construction of the N_k proceeds as follows. Let G denote one of the groups $Sp(n)$, $SO(2n+1)$, or $Spin(2n+1)$ and let τ be an automorphic involution on G with fixed point set K . Consider the hyperkähler quotient Y^- of W_k^- by K (with zero-level set of the moment map). Let (T_0, T_1, T_2, T_3) be a solution to Nahm's equations corresponding to a point in Y^- . Modulo gauge transformations we can assume that $T_0(1) = 0$. We can then extend this solution to a solution to Nahm's equations on $(0, 2)$ by setting

$$(2.1) \quad T_i(2-t) = -\tau(T_i(t)), \quad i = 0, 1, 2, 3.$$

This solution has the boundary behaviour of a solution in N_k and we can describe the moduli space Y of such extended solutions as the space of solutions on $(0, 2)$ having the correct poles and residues at $t = 0, 2$ and satisfying (2.1), modulo G -valued gauge transformations $g(t)$ such that

$$(2.2) \quad g(0) = g(2) = 1, \quad g(2-t) = \tau(g(t)), \quad t \in [0, 2].$$

The map $Y^- \rightarrow Y$ is a triholomorphic homothety with factor 2. For dimensional reasons Y^- (and consequently Y) is empty unless $\mathfrak{k} = \mathfrak{u}(n)$ for $k = 2n$ or $\mathfrak{k} = \mathfrak{so}(n) \oplus \mathfrak{so}(n+1)$ for $k = 2n+1$. Thus there are the following three possibilities for the symmetric pair (G, K) :

- (i) $k = 2n$, $G = Sp(n)$ and $K = U(n)$;
- (ii) $k = 2n+1$, $G = SO(2n+1)$ and $K = S(O(n) \times O(n+1))$;
- (iii) $k = 2n+1$, $G = Spin(2n+1)$ and K is the diagonal double cover of $SO(n) \times SO(n+1)$ (i.e. $K = Spin(n) \times Spin(n+1)/\{(1, 1), (-1, -1)\}$).

An easy computation shows that in each case $\dim Y = \dim N_k$. Moreover, the natural map $Y \rightarrow N_k$ is an isometric (and triholomorphic) immersion, and since

both N_k and Y are complete ([3, Thm. A.1]), this map must be a covering. Thus it follows from Proposition 1.5 that Y is isometric to N_k in cases (i) and (ii), while in case (iii) Y is the universal cover of N_{2n+1} .

The above construction allows us easily to describe a family of deformations of N_{2n} . Indeed, the Lie algebra $\mathfrak{k} = \mathfrak{u}(n)$ has a nontrivial centre and, therefore, we can take hyperkähler quotients of W_{2n}^- by K at nonzero level sets of the hyperkähler moment map. This produces a 3-parameter family of hyperkähler deformations of N_{2n} . Arguments analogous to those in [3] show that these are the natural hyperkähler metrics on the transverse Hilbert schemes of points on Dancer's deformations of the D_1 -surface.

Remark 2.1. As already mentioned in Remark 1.7, N_{2n} admits a free action of \mathbb{Z}_2 for any n . This action is also isometric and triholomorphic and, hence, N_{2n}/\mathbb{Z}_2 is a hyperkähler manifold. This manifold can be described in the same way as N_{2n} but with $G_{2n} = \mathbb{P}Sp(n)$ rather than $Sp(n)$.

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